Granular Mereotopology: A First Sketch

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Abstract. Mereotopology aims at a reconstruction of notions of set topology in mereological universa. Because of foundational differences between set theory and mereology, most notably, the absence of points in the latter, the rendering of notions of topology in mereology faces serious difficulties. On the other hand, some of those notions, e.g., the notion of a boundary, belong in the canon of the most important notions of mereotopology, because of applications in, e.g., geographic information systems. Rough mereology allows for a formal theory of knowledge granulation, and, granules may serve as approximations to open sets, hence, it is reasonable to explore the possibility of their usage in buildup of mereotopological constructs. This work is segmented into sections on mereology, rough mereology, granule theory, mereotopology.
Keywords: spatial reasoning, mereotopology, rough mereology, boundary, open set.

1 Standard Mereology

Under the term *Standard Mereology* we understand the theory of parts constructed by Stanislas Leśniewski, cf. [8], [10], [13]. Given some collection (a universe), say U, of things, a relation of part on them is a binary relation *part* which is required to be

M1 *Irreflexive*: *For each* $x \in U$ *it is not true that* $part(x, x)$

M2 *Transitive*: *For each triple* x, y, z *of things in* U, *if* $part(x, y)$ *and* $part(y, z)$, *then* $part(x, z)$

Fig. 1 shows the chessboard with parts being white and black squares.

The relation of *part* gives rise to the relation of an *ingredient*, *ingr*, defined as

$$ingr(x, y) \Leftrightarrow \pi(x, y) \vee x = y. \tag{1}$$

Clearly, the relation of an ingredient is a partial order on things.

We formulate the third axiom of Standard Mereology which does involve the notion of an ingredient. Before it, we introduce a property of things. For things x, y, we let,

Fig. 1. White and black squares as parts of the chessboard

$I(x,y)$: *For each thing* t *such that* $ingr(t,x)$, *there exist things* w,z *such that* $ingr(w,t), ingr(w,z), ingr(z,y)$

Now, we can state an axiom.

M3 (*Inference Rule*) *For each pair of things* x,y, the property $I(x,y)$ implies that $ingr(x,y)$

The predicate of *overlap*, Ov in symbols, is defined by means of

$$Ov(x,y) \Leftrightarrow \ \exists z.ingr(z,x) \wedge \ ingr(z,y). \tag{2}$$

1.1 The class operator

Aggregation of things into a composite thing is done in set theory by means of the union of sets operator. Its counterpart, and a generalization, in mereology, is the class operator. For a non–empty property Φ of things, the *class of* Φ, denoted $Cls\Phi$, is defined by the conditions

C1 *If* $\Phi(x)$, *then* $ingr(x, Cls\Phi)$

C2 *If* $ingr(x, Cls\Phi)$, *then there exists* z *such that* $\Phi(z)$ *and* $I(x,z)$

In plain language, the class of Φ collects in an individual object all objects satisfying the property Φ.
The existence of classes is guaranteed by an axiom.
M4 *For each non–vacuous property* Φ *there exists a class* $Cls\Phi$
The uniqueness of the class follows by M3.
In Fig. 1, we can discuss the class of white squares, the class of black squares, or, the class of occupied squares.

Example 1. 1. The strict inclusion $\subset$ on sets is a part relation. The corresponding ingredient relation is the inclusion $\subseteq$. The overlap relation is the non–empty intersection. For a non–vacuous family F of sets, the class $ClsF$ is the union $\bigcup F$;

2. For reals in the interval $[0, 1]$, the strict order $<$ is a part relation and the corresponding ingredient relation is the weak order $\leq$. Any two reals overlap; for a set $F \subseteq [0, 1]$, the class of F is $supF$.

The notion opposite to that of overlap is the notion of *disjointness*: its symbol is $extr$, and, for things x, y,

$$extr(x, y) \Leftrightarrow \text{it is not true that } Ov(x, y). \tag{3}$$

The notion of a *complement* to an object, relative to another object, is rendered as a ternary predicate $comp(x, y, z)$, [8], par. 14, Def. IX, to be read:'x is the complement to y relative to z', and it is defined by means of the following requirements,

1. $x = ClsEXTR(y, z)$;
2. $ingr(y, z)$,

where $EXTR(y, z)(t)$ holds if and only if $ingr(t, z)$ and $extr(t, y)$.

This definition implies that the notion of a complement is valid only when there exists an ingredient of z exterior to y.

The notion of a class has been extensively studied motivated by its fundamental importance for foundations of mathematics, logics and mereology, cf., e.g., Lewis [9].

For the property $Ind(x) \Leftrightarrow\ ingr(x, x)$, one calls the class $ClsInd$, *the universe*, in symbols V,cf.,[8], par. 12, Def. VII. The complement with respect to the universe of a thing serves as the complement in algebraic sense.

We let for an object x,

$$-x = ClsEXTR(x, V). \tag{4}$$

It follows that

1. $-(-x) = x$ *for each object* x;
2. $-V$ *does not exist.*

In Fig. 1, the complement to the class of white squares is the class of black squares (we assume that classes of squares are ingredients of the chessboard as well). The operator $-x$ can be a candidate for the Boolean complement in a structure of a Boolean algebra within Mereology, constructed in [18], and anticipated in [17]; in this respect, cf., [5]. This algebra will be obviously rid of the null element, as the empty object is not allowed in Mereology , and the meet of two objects will be possible only when these objects overlap. Under this caveat, the construction of Boolean operators of join and meet proceeds as in [18].

We define the Boolean sum $x + y$ by letting

$$x + y = Cls(t : ingr(t, x) \vee ingr(t, y)). \tag{5}$$

In Fig. 2, we give an example of the sum which is the full moon as the sum of the two quarters: 4th and 1st ('halves').

Fig. 2. The 4th quarter of the moon =x; the 1st quarter of the moon =y; the full moon= x+y

The product $x \cdot y$, cf., [18] is defined in a parallel way,

$$\text{If } Ov(x, y) \text{ then } x \cdot y = Cls(t : ingr(t, x) \wedge ingr(t, y)). \quad (6)$$

Operators $+, \cdot, -$ and the unit V introduce the structure of a complete Boolean algebra without the null element, cf., [18], [13].

An often invoked example of a mereological universe is the collection ROM_n of regular open sets in the Euclidean space E^n; we recall that an open set A is *regular open* when

$$A = IntClA, \quad (7)$$

where Int, Cl are , respectively, the interior and the closure operators of topology, see, e.g., [10], Ch.2. In this universe, mereological notions are rendered as

1. $ingr(A, B) \Leftrightarrow A \subseteq B$;
2. $part(A, B) \Leftrightarrow A \subset B$;
3. $Ov(A, B) \Leftrightarrow A \cap B \neq \emptyset$;
4. $A \cdot B = A \cap B$;
5. $-A = R^n \setminus ClA$;
6. $A + B = A \cup B$.

2 Rough Mereology

Rough Mereology, cf., , [10], [11], [12], introduces the notion of a part to a degree, $\mu(x, y, r)$ read '*x is a part in y to a degree of r*' with requirements

RM1 $\mu(x, y, 1) \Leftrightarrow ingr(x, y)$

RM2 $\mu(x, y, 1) \wedge \mu(z, x, r) \Rightarrow \mu(z, y, r)$

RM3 $\mu(x, y, r) \wedge s \leq r \Rightarrow \mu(x, y, s)$

where *ingr* is the ingredient relation in an a priori assumed Mereology.

The relation μ called *a rough inclusion* in [11] can be induced in some ways from t–norms, for t–norms, see, e.g., [6], [10], Ch. 4.

2.1 Rough inclusions from residua of continuous t–norms

In the first case, for a continuous t–norm t, cf., e.g., [6], [10], Ch. 4, Ch. 6.2., the *residual implication* $x \Rightarrow_t y$ defined as

$$x \Rightarrow_t y = max\{r : t(x,r) \leq y\}, \tag{8}$$

yields the rough inclusion

$$\mu_t(x,y,r) \Leftrightarrow x \Rightarrow_t y \geq r. \tag{9}$$

2.2 Rough inclusions from archimedean t–norms

In the other case, for the *t–norm of Łukasiewicz*,

$$t_L(x,y) = max\{0, x+y-1\}, \tag{10}$$

or, the *product* t–norm,

$$t_P(x,y) = xy, \tag{11}$$

see, e.g., [10], Ch. 4, which admit representations,

$$t_L(x,y) = g_L(f_L(x)+f_L(y)), t_P(x,y) = g_P(f_P(x)+f_P(y)) \tag{12}$$

with

$$g_L(x) = 1-x = f_L(x), g_P(x) = exp(-x), f(x) = -lnx, \tag{13}$$

cf., [6], [10], Ch. 4, one defines the rough inclusion

$$\mu^L(x,y,r) \Leftrightarrow g_L(|x-y|) \geq r, \tag{14}$$

respectively,

$$\mu^P(x,y,r) \Leftrightarrow g_P(|x-y|) \geq r. \tag{15}$$

The last formula can be transferred to the realm of finite sets, with g either g_L or g_P, as

$$\mu^L_s(X,Y) = g(\frac{|X \triangle Y|}{|X|}) = \frac{|X \cap Y|}{|X|}, \tag{16}$$

to the case of bounded measurable sets in E^n as

$$\mu^L_G(X,Y) = g(\frac{||X \triangle Y||}{||X||}) = \frac{||X \cap Y||}{||X||}, \tag{17}$$

where $a \triangle b$ denotes the symmetric difference of a, b, $|a|$ is the cardinality of a, and, $||a||$ is the measure (area) of a.

2.3 Transitivity of rough inclusions

An important property of rough inclusions is the *transitivity property*. For rough inclusions of the form μ^t with t being L or P, as well as for rough inclusions of the form μ_t this property has the form, see Polkowski [10], Props. 6.7, 6.16,

$$\mu^t(x, y, r) \wedge \mu^t(y, z, s) \Rightarrow \mu^t(x, z, t(r, s)). \tag{18}$$

In case of rough inclusions of the form μ_t, it becomes,

$$\mu_t(x, y, r) \wedge \mu_t(y, z, s) \Rightarrow \mu_t(x, z, t(r, s)). \tag{19}$$

3 Granules as weakly open sets in rough mereology

We begin our study of mereotopology in a rough mereological universe U with a given rough inclusion μ. In order to introduce topological structures, we first introduce a mechanism of granulation in U. For a thing x in U and a real number r in the interval $[0, 1]$, we define the *granule* $g(x, r, \mu)$, about x of radius r, as

$$g(x, r, \mu) \text{ is } ClsM(x, r, \mu), \tag{20}$$

where

$$M(x, r, \mu)(y) \Leftrightarrow \mu(y, x, r). \tag{21}$$

Granules can be characterized in terms of rough inclusions as follows, see Polkowski [10], Ch. 7, Props. 7.1, 7.2.

Proposition 1. *For granules induced by rough inclusions of the form μ^t as well as for granules induced by the rough inclusion μ_M, we have for each pair x, y of things, $ingr(y, g(x, r, \mu))$ if and only if $\mu(y, x, r)$.*

For granules induced by rough inclusions μ_L, μ_P, the situation is more complicated, see Polkowski [10], 7.3.

3.1 Open sets

We apply the granules to define neighborhoods of things in U. To this end, we define a property $N(x, r, \mu)$ by letting,

$$N(x, r, \mu)(y) \Leftrightarrow \exists s > r.\mu(y, x, s). \tag{22}$$

The *neighborhood* $n(x, r, \mu)$ of a thing x of radius r relative to μ is defined as

$$n(x, r, \mu) \text{ is } ClsN(x, r, \mu). \tag{23}$$

The neighborhood system has properties of open sets, viz., see [10], Ch. 7,

1. If $ingr(y, n(x, r, \mu))$, then $\exists s.ingr(n(y, s, \mu), n(x, r, \mu))$;
2. If $s > r$, then $ingr(n(x, s, \mu), n(x, r, \mu))$;

3. If $ingr(y, n(x, r, \mu))$ and $ingr(y, n(z, s, \mu))$, then

$$\exists q.ingr(n(z, sq, \mu), n(x, r, \mu)) \text{ and } ingr(n(z, q, \mu), n(y, s, \mu)).$$

We define an open set as a collection of neighborhoods; the predicate $open(F)$ is therefore defined as,

$$open(F) \Leftrightarrow \forall z.[z \in F \Leftrightarrow z\ is\ n(x, r, \mu)\ for\ some\ x, r\ ]. \tag{24}$$

It is now possible to define open things as classes of open collections,

$$open(x) \Leftrightarrow \exists F.open(F) \wedge x\ is\ ClsF. \tag{25}$$

Closed things are defined as complements to open things,

$$closed(x) \Leftrightarrow open(-x). \tag{26}$$

We may need as well the notion of a closed collection, as the complement to an open collection,

$$closed(F) \Leftrightarrow open(-F), \tag{27}$$

where, clearly, the complement $-F$ is the collection obtained by applying the mereological complement $-$ to each member of F.

4 Boundaries

The practical importance of boundaries stems from their role as separating regions among areas of interest like roads, rivers, fields, forests etc., and this causes the theoretical interest in them. The notion of a boundary has been studied in philosophy, mathematics, computer science by means of mereology. Mathematics resolved the problem of boundaries by topological notion of the boundary (frontier) BdX of a set X in a topological space (U, τ) which was defined as

$$BdX = ClX \setminus IntX, \tag{28}$$

i.e., any point $x \in U$ satisfies

$$x \in BdX \Leftrightarrow \forall P.P\ open \wedge x \in P \Rightarrow P \cap X \neq \emptyset \neq P \cap (U \setminus X).$$

It is evident from this definition that the notion of the boundary of X involves in the symmetrical way the complement:

$$BdX = Bd(U \setminus X). \tag{29}$$

It also follows that the notion of a boundary is of infinitesimal character as detecting whether $x \in BdX$ involves neighborhoods of x of arbitrarily small size.

Philosophers noticed this duality of boundaries between things and their complements and went even further, in the extreme cases, assigning the boundary character to any thing by considering it as a potential boundary (the phenomenon of *plerosis*, e.g., a point in the open disc can be the end point of any radius from it to the perimeter of the disc, a fortiori, in the boundary of continuum many segments. Moreover, e.g., the perimeter of the planar disc, considered, e.g., in 3D space, can be the boundary of continuum many bubbles spanned on the perimeter , see [2], [3], [15]).

4.1 Mereoboundaries

Topological definition of boundary led Smith [14] toward a scheme for defining mereoboundaries. First, He proposes an axiomatic introduction of open sets as *interior parts*, IP in symbols. In this context, the notion of *straddling*, Str in symbols, is defined as,

$$Str(x,y) \Leftrightarrow [\forall z.IP(x,z) \Rightarrow Ov(z,y) \wedge Ov(z,-y)]. \tag{30}$$

The notion of a *boundary part* is introduced in Smith [14] by means of an auxiliary predicate

$$B(x,y) \Leftrightarrow \forall z.ingr(z,x) \Rightarrow Str(z,y). \tag{31}$$

Boundary $Bd(y)$ of a thing y is defined as

$$Bdy(y) = Cls\{x : B(x,y))\}. \tag{32}$$

It is a straightforward task to verify that in the space ROM_n of regular open sets, each set x is an interior set of each of its supersets and requirements for IP are fulfilled, $Str(x,y)$ is satisfied in case $Ov(x,y) \wedge Ov(x,-y)$ and $B(x,y)$ is satisfied for no x, y hence the boundary is not defined being empty. The reason is a too liberal definition of straddling, allowing mere ingredients of a given thing x.

4.2 Granular mereoboundaries

For this reason, we re–model the approach by Smith in [14] by allowing granular neighborhoods as open things, a fortiori interior parts, and by restricting interior parts to granular neighborhoods about the same thing. In detail, our approach presents itself as follows.

We say that a granular neighborhood $n(x,r,\mu)$ *granular straddles* a thing y if and only if the following property $GStr(x,r,y)$ holds,

$$GStr(x,r,y) \Leftrightarrow \forall s \in (r,1).Ov(n(x,s,\mu),y) \wedge Ov(n(x,s,\mu),-y). \tag{33}$$

Let us observe that the notion of granular straddling is *downward hereditary* in the sense that

$$GStr(x,r,y) \wedge s > r \Rightarrow GStr(x,s,y). \tag{34}$$

Also, it is manifest that this notion is *upward hereditary*, i.e.,

$$GStr(x, r, y) \Rightarrow \forall s < r.GStr(x, s, y). \quad (35)$$

With each granule $g(x, r, \mu)$, we associate the collection $GN(x, \mu) = \{n(x, s, \mu) : s \in (0, 1)\}$, which we call the $x - ultrafilter\ base$. We say that an x-ultrafilter base $GN(x, \mu)$ *granular straddles* a thing y if and only if there exists an $s \in (0, 1)$ such that $GStr(x, s, \mu), y)$ holds. We denote this fact with the symbol $B(x, y)$.

We regard the collection $GN(x, \mu)$ as a *point at infinity* and, according to the topological nature of boundary, we assign to the thing x such that the x-ultrafilter base $GN(x, \mu)$ granular straddles a thing y this point at infinity as the boundary point of y. Hence, we define the boundary of y, in symbols Bdy, as the collection of those points,

$$Bdy \text{ is } \{x : \ B(x, y)\}. \quad (36)$$

Boundaries defined in this way are $ingr - upward - hereditary$ in the sense,

$$ingr(z, x) \wedge B(z, y) \Rightarrow B(x, y). \quad (37)$$

The proof follows from definitions by M3 and transitivity of the applied rough inclusion. In view of the correspondence between things and ultrafilter bases, we may say that the thing x is a boundary point of the thing y in case the x–ultrafilter base granular straddles y. This approach does satisfy philosophical postulates about boundary like

1. The boundary of a thing may not belong in the universe of considered things; in other words, the boundary is of different topological type then the thing;
2. in order to preserve the typology of the boundary one has to preserve its infinitesimal character;
3. the boundary of a thing may be a boundary of a plethora of other things , in particular, by necessity, it has to be the boundary of each complement to the thing.

Let us observe that the set–theoretic complement to Bdy is open as it is the collection,

$$\{z : \exists s.ingr(n(z, s, \mu), y) \vee ingr(n(z, s, \mu), -y)\}, \quad (38)$$

hence, Bdy is a closed collection for each thing y.

It is a straightforward task to check that in the space ROM_n, for a regular open set A, the granular boundary is defined by

$$B(Z, A) \Leftrightarrow ClZ \cap A \neq \emptyset. \quad (39)$$

5 Conclusion

We admit an infinitesimal nature of boundaries along with the fact that their nature is distinct from the nature of things they bound, like it happens to closed nowhere dense boundaries of regular open sets, and we represent them by means of ultrafilters constructed in the meta–space of collections of things. We have aimed at giving a definition of boundary in purely mereological terms, without any resort to augmentations which are necessary for a more exact description, like geographic directions, notions of touching, contact, beacons, in a word many other than mereological primitive notions, see, e.g., [1], [7].

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